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Problem 11279. Two test mass beads are sliding along a vertical circular track under (Newtonian) constant gravity, without friction. The first bead M_1 starts from the highest point on the circle, at some nonzero velocity. Some time later the second bead M_2 starts from the same position at the top of the circle and with the same initial velocity as M_1 had. Prove that there is a circle to which the line through the current positions of M_1 and M_2 is always tangent, and find the center and radius of that circle in terms of the original circle.

First, we define a set of coordinates where the y-axis is up and the circle lies in the x-y plane. We measure angles counterclockwise from the x-axis, and pick our coordinates so that the beads move counterclockwise. We call the radius of the circle R, and characterize the beads as having mechanical energy E when the zero of gravitational potential energy is placed at the center of the circle.

We define the bead M_1 to be the bead which is higher when the beads are vertically aligned and both moving up. At the point that we make this definition, the coordinate θ_1 of M_1 is some angle we can call β, and $\theta_2 = -\beta$. We can see by symmetry that it will take the same amount of time for M_1 to move from the point β to the point $\pi + \beta$ as it will for M_2 to move from $-\beta$ to $\pi - \beta$ (to see this first note that were M_2 to move clockwise from $\pi - \beta$ to $-\beta$ it would clearly take the same amount of time, and then recall that since velocity depends only on position on the circle, any trip will take the same amount of time in either direction). This means that the center of the small circle must lie on the y-axis.

Indeed, the circle has radius $R\cos\beta$ and is centered at the point $(0, kR)$ where k is a solution to the equation $k^2 - \cos^2\beta + 1 = \frac{2kE}{mgR}$. The derivation of this fact follows.

If we try to compute the positions of the beads directly, we define $\omega_i = \dot{\theta}_i$ and note that the energy (which is constant) is given by $E = \frac{1}{2}mR^2\omega_i^2 + mgR\sin\theta_i$. Solving this for ω_i we get

$$\omega_i = \sqrt{\frac{2E}{mR^2} - \frac{2g\sin\theta_i}{R}} = \sqrt{\frac{2g}{R}}\sqrt{\frac{E}{mgR} - \sin\theta_i}$$

If we define $y = \frac{E}{mgR}$ then $\omega_i = \sqrt{\frac{2g}{R}}\sqrt{y - \sin\theta_i}$, which cannot be integrated without resorting to elliptic integrals.

Now consider what it means for the line between the two beads to be tangent to a circle of radius rR (recall that $r = \cos\beta$) with center $(0, kR)$. A line from $(R\cos\theta_1, R\sin\theta_1)$ to $(R\cos\theta_2, R\sin\theta_2)$ will touch the circle at a point $(rR\cos\phi, kR + rR\sin\phi)$, where $\tan\phi$ is the slope of a line perpendicular to the line connecting M_1 and M_2. That is:

$$\frac{R\sin\theta_i - kR - rR\sin\phi}{R\cos\theta_i - rR\cos\phi} = -\frac{\cos\phi}{\sin\phi}, \ i = 1, 2$$

This can be rewritten as

$$\sin\theta_i \sin\phi - k\sin\phi - r\sin^2\phi = -\cos\theta_i\cos\phi + r\cos^2\phi$$

which reduces to $\cos(\theta_i - \phi) = r + k\sin\phi$. If we define $\alpha = \cos^{-1}(r + k\sin\phi)$ then $\theta_1 = \phi + \alpha$ and $\theta_2 = \phi - \alpha$.

This means we need to show that if when we define $\phi = \frac{\theta_1+\theta_2}{2}$ and $\alpha = \frac{\theta_1-\theta_2}{2}$ and it is true that $\cos\alpha = r + k\sin\phi$ at some time t then this last equation will continue to hold true for all later t. This means we need to show that the time derivative of this equation holds, that is, $-\dot{\alpha}\sin\alpha = k\dot{\phi}\cos\phi$. Rewriting this using the definitions of α and ϕ we get

$$-\frac{\sin\alpha}{2}\sqrt{\frac{2g}{R}}\left(\sqrt{y-\sin\theta_1} - \sqrt{y-\sin\theta_2}\right) = k\frac{\cos\phi}{2}\sqrt{\frac{2g}{R}}\left(\sqrt{y-\sin\theta_1} + \sqrt{y-\sin\theta_2}\right)$$

Multiplying through by $\dot{\alpha}$ and simplifying we get

$$-\sin\alpha\left(2y - \sin\theta_1 - \sin\theta_2 - 2\sqrt{(y-\sin\theta_1)(y-\sin\theta_2)}\right) = k\cos\phi(\sin\theta_2 - \sin -\theta_1)$$

Replacing θ_i with ϕ and α (and dividing by $2\sin\alpha$) gives

$$y - \sin\phi\cos\alpha - \sqrt{(y-\sin\phi\cos\alpha)^2 - \cos^2\phi\sin^2\alpha} = k\cos^2\phi$$

Next we use $\cos\alpha = r + k\sin\phi$ to write this equation exclusively in terms of ϕ. This substitution plus a little rearrangement gives

$$y - k - r\sin\phi = \sqrt{y^2 + r^2 + k^2\sin^2\phi + 2kr\sin\phi - 2yr\sin\phi - 2ky\sin^2\phi - \cos^2\phi}$$

Squaring both sides and simplifying we get

$$k^2 + r^2\sin^2\phi - 2ky = r^2 + k^2\sin^2\phi - 2ky\sin^2\phi - \cos^2\phi$$

which simplifies to $k^2 - r^2 + 1 = 2ky$. All the steps in the derivation are reversible (one consequence of the final equation is to remove the sign ambiguity when taking the square root that must be taken when working backwards) so we could just have easily have started from $k^2 - r^2 + 1 = 2ky$ and concluded that $-\dot{\alpha}\sin\alpha = k\dot{\phi}\cos\phi$, and thus that, as stated above, the line from M_1 to M_2 is always tangent to a circle of radius rR centered at $(0, kR)$.