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Problem 11297. For positive a, b, and c, let

$$E(a,b,c)=\frac{a^2b^2c^2-64}{(a+1)(b+1)(c+1)-27}$$

Find the minimum value of $E(a,b,c)$ on the set D consisting of all positive triples (a,b,c) other than $(2,2,2)$ at which $abc=a+b+c+2$.

The minimum is $\frac{23+\sqrt{17}}{8}$. It occurs at $(\frac{3+\sqrt{17}}{2},\frac{3+\sqrt{17}}{2},\frac{-1+\sqrt{17}}{4})$.

To see this, start with a variable transformation. Let $p=abc$ and let $q=ab$. The set D now consists of triples that satisfy $p=a+\frac{q}{a}+\frac{p}{q}+2$. If we note that $(a+1)(b+1)(c+1)=abc+ab+ac+bc+a+b+c+1$ and make appropriate substitutions we can derive

$$E'(a,q,p)=\frac{p^2-64}{2p+q+\frac{ap}{q}+\frac{p}{a}-28}$$

We proceed by the method of Lagrange multipliers:

$$F(a,p,q,\lambda)=\frac{p^2-64}{2p+q+\frac{ap}{q}+\frac{p}{a}-28}-\lambda\left(p-a-\frac{q}{a}-\frac{p}{q}-2\right)$$

$$\frac{\partial F}{\partial a}=-\frac{(p^2-64)(\frac{p}{q}-\frac{p}{a^2})}{(2p+q+\frac{ap}{q}+\frac{p}{a}-28)^2}-\lambda(-1+\frac{q}{a^2})=(\frac{q}{a^2}-1)\left(\lambda+\frac{p}{q}\left(\frac{p^2-64}{(2p+q+\frac{ap}{q}+\frac{p}{a}-28)^2}\right)\right)=0$$

$$\frac{\partial F}{\partial q}=-\frac{(p^2-64)(1-\frac{ap}{q^2})}{(2p+q+\frac{ap}{q}+\frac{p}{a}-28)^2}-\lambda(-\frac{1}{a}-\frac{p}{q^2})=(\frac{1}{a^2}-\frac{p}{q^2})\left(\lambda+a\left(\frac{p^2-64}{(2p+q+\frac{ap}{q}+\frac{p}{a}-28)^2}\right)\right)=0$$

From the last two equations we can conclude that at least one of the following is true: $\frac{q}{a^2}=1$ or $\frac{1}{a}=\frac{p}{q^2}$ or $\frac{p}{q}=a$. These three possibilities imply, respectively, that $a=b$, $b=c$, or $a=c$. Thus for a minimum to occur at least two of the coordinates must be the same.

We will assume $a=c$. Then our conditions become $a^2b=2a+b+2$ (It is also possible that $a=c=-1$ and b is any real number, but this violates the requirement that all three coordinates be positive). Solving this for b gives $b=\frac{2}{a-1}$. Thus we can define

$$e(a)=E(a,\frac{2}{a-1},a)=\frac{\frac{4a^4}{(a-1)^2}-64}{\frac{(a+1)^3}{(a-1)}-27}=\frac{4a^4-64(a-1)^2}{(a+1)^3(a-1)-27(a-1)^2}$$

$$e(a)=\frac{4a^4-64a^2+128a-64}{a^4+2a^3-27a^2+52a-28}=4\frac{x^2+4x-4}{x^2+6x-7}$$

$$\frac{de}{da}=4\frac{2x^2-6x-4}{(x^2+6x-7)^2}$$

Thus $\frac{de}{da} = 0$ when $a = \dfrac{3 \pm \sqrt{17}}{2}$ and so we find that the minimum occurs at $(\frac{3+\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{4}, \frac{3+\sqrt{17}}{2})$.

Strictly speaking at this point we have only demonstrated that this is a critical point, not a minimum. However, since this is the only critical point in the set D, the minimum of the function must be either at this point or at the boundary of the set D. To see that the minimum does not occur at the boundary, start by fixing $a = A$. Then $b = \frac{A+c+2}{Ac-1}$. The edges of the set D (with $a = A$) are at $c = \frac{1}{A}$ and $c = \infty$. Now simply note:

$$\lim_{c \to \frac{1}{A}} \frac{A^2c^2(\frac{A+c+2}{Ac-1})^2 - 64}{(A+1)(c+1)(\frac{A+c+Ac+1}{Ac-1}) - 27} = \lim_{c \to \frac{1}{A}} \frac{A^2c^2\frac{(A+c+2)^2}{(Ac-1)} - 64(Ac-1)}{(A+1)^2(c+1)^2 - 27(Ac-1)} = \infty$$

$$\lim_{c \to \infty} b = \frac{1}{A} \therefore \lim_{c \to \infty} \frac{A^2b^2c^2 - 64}{(A+1)(b+1)(c+1) - 27} = \lim_{c \to \infty} \frac{c^2 - 64}{(A+1)(\frac{1}{A}+1)(c+1) - 27} = \infty$$

The facts that we chose A arbitrarily and that we can permute a, b, and c mean that this implies that $E(a, b, c)$ approaches infinity at any boundary of the set D and therefore it clearly cannot have a minimum there. Thus the minimum must occur at the critical point we found.