Problem: A metal bar five feet in length hangs from a level ceiling by two light, unstretchable ropes, one three feet long and the other four feet long. The two ropes attach to the ceiling ten feet apart. Determine the angle between the shorter rope and the ceiling.

Solution: This turns out to be rather tricky. We start by noting that the system will tend to minimize its potential energy, and thus that this is equivalent to maximizing the sum to the distances from the ceiling to each end of the bar. That is, we are trying to maximize $x+y$ subject to

$$(9-x^2)^{\frac{1}{2}}+(1-y^2)^{\frac{1}{2}}+(25-(x-y)^2)^{\frac{1}{2}}-10=0$$

.

Using the method of Lagrange multipliers, we take

$$F(x,y,\lambda)=x+y+\lambda((9-x^2)^{\frac{1}{2}}+(1-y^2)^{\frac{1}{2}}+(25-(x-y)^2)^{\frac{1}{2}}-10)$$

and set all the partial derivatives to zero. The derivative with respect to λ simply gives the condition in the first equation. Taking derivatives with respect to x and y, solving each equation for λ and doing some tedious algebra gives us

$$x(9-x^2)^{-\frac{1}{2}}-y(16-y^2)^{-\frac{1}{2}}+2(x-y)(25-(x-y)^2)^{-\frac{1}{2}}=0$$

Now we have two equations in two unknowns, and we could use the (absurdly complex) quartic formula on the first equation (after expanding it to get rid of the square roots) and then plug into the second equation to get an equation in one variable. But it wouldn't be at all fun; just turning that into a quartic equation involved coefficients that exceed one million. So instead we take full derivatives with respect to x of each of our two equations, and solve for $\frac{dy}{dx}$. We get:

$$\frac{dy}{dx}=\frac{x(9-x^2)^{-\frac{1}{2}}+(x-y)(25-(x-y)^2)^{-\frac{1}{2}}}{(x-y)(25-(x-y)^2)^{-\frac{1}{2}}-y(16-y^2)^{-\frac{1}{2}}}$$

and

$$\frac{dy}{dx}=\frac{9(9-x^2)^{-\frac{3}{2}}+50(25-(x-y)^2)^{-\frac{3}{2}}}{16(16-y^2)^{-\frac{3}{2}}+50(25-(x-y)^2)^{-\frac{3}{2}}}$$

We can now use a modified Newton's method to find the solutions. We pick a value for x_0. Then we solve each of the two original equations for y (call these values y_1 and y_2. We use these values of y to calculate $\frac{dy}{dx}$ in each of the second two equations (call these results $\frac{dy_1}{dx}$ and $\frac{dy_2}{dx}$. Then we let

$$x_{n+1}=x_n+\frac{y_1-y_2}{\frac{dy_2}{dx}-\frac{dy_1}{dx}}$$

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(If you approximate each implicitly defined function as a line near (x,y) then this is the x value where the two lines intersect). It takes about thirty iterations on a TI-84 calculator (using the built in function to solve for y each time) to get an answer that has converged to nine decimal places.

We find that $x = 2.200995639$ and $y = 2.664868869$, so the angle we are looking for becomes $47.194548°$.